Analysis I Lecture 18

Last time:

- Continuous functions on intervals.

Here interval I is a set of the form: $[a,b]$, $[a,b)$, $(a,b]$, (a,b), $(-\infty, a)$, $(-\infty, a]$, $[b, +\infty)$, $(b, +\infty)$.

$$\mathbb{R} = (-\infty, +\infty)$$

Theorem (Intermediate value theorem)

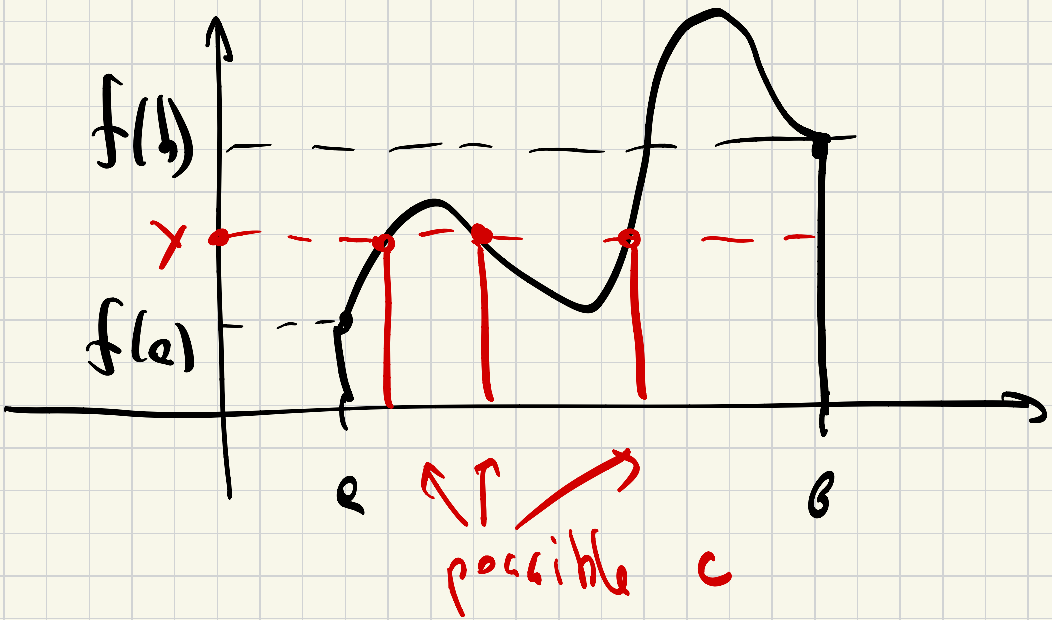
Theorem 1.66 in Notes on functions

Let $f \in C^0(I)$ and $a, b \in I$

← the set of cont. functions on I

then for every y between $f(a)$ and $f(b)$

there is $c \in [a, b]$ s.t. $f(c) = y$.



Corollary (Banach fixed point thm) Corollary 1.65

Let $f : [a,b] \to [a,b]$ be continuous

then there exists $c \in [a,b]$ s.t.

$$f(c) = c.$$

In other words

$f \in C^0(I)$ and $I = [a,b]$

$a \le f(x) \le b \quad \forall x \in [a,b]$

Example $f(x) = |\sin(x)| + 1$

Example $\boxed{f(x) = |\sin(x)| + 1}$

Notice $f(x): [1,2] \to [1,2]$

Indeed $1 \leq f(x) \leq 2$ for every $x \in [1,2]$ since $0 \leq |\sin x| \leq 1$.

$\Rightarrow$ By Banach fixed point theorem $\exists\, x \in [1,2]$ s.t. $|\sin(x)| + 1 = x$.

Theorem Let I be an interval and $f \in C^0(I)$, then

1) The image of f is an interval.

2) If f is strictly monotone, and I is an open interval, then $f(I)$ is open interval

Corollary 1.72

3) If $I = [a,b]$ closed then $\min_{x \in I} f$ and $\max_{x \in I} f$ exist and $f(I) = \left[\min_{x \in I} f, \max_{x \in I} f\right]$

Theorem 1.63

Examples: 1) $f(x) = \sin(x)$

$f: \mathbb{R} \to \mathbb{R}$ the image of f $[-1, 1]$

Described by theorem part 1.

2) $f: (0, 1) \to \mathbb{R}$ $f(x) = \frac{1}{x}$

defined on open interval + f stricly monotone

$f(x) < f(y)$ for any $x > y \in (0,1)$

$\mathrm{Im}(f) = (1, +\infty)$ Parts 1, 2 of thm.

3) $f : (-1,1) \to \mathbb{R}$ $\quad f(x) = x^2$

$f(x)$ is not monotone

ony Part 1 of thm is applicable

And $\text{Im}(f) = [0, 1)$

Notice $0 \in (-1,1)$ and

$0^2 = 0 \Rightarrow 0 \in \text{Im}(f)$

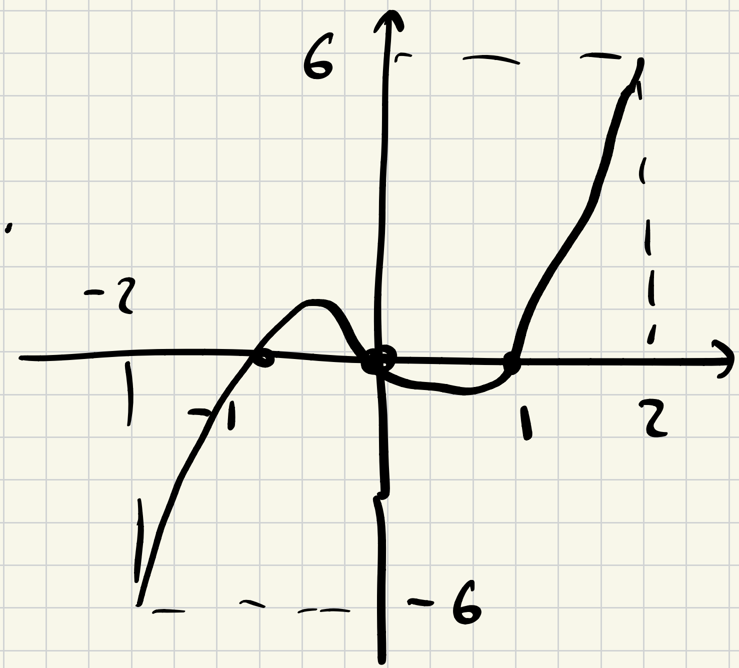
But $1 \notin \text{Im}(f)$ since $|x| < 1$ for $x \in (-1,1)$

4) $f : (-2, 2) \to \mathbb{R}$ $\quad f(x) = x^3 - x$

Notice that $f(x)$ is <u>not</u> strictly monotone

However in this case

$$\mathrm{Im}(f) = (-6, 6).$$



Today

- Continuous functions on intervals;
- Reminder on injective / surjective / bijective functions. As well as inverse functions.
- Derivatives.

Recall

Injective / Surjective / bijective functions

Def We say that a function

$f: D \to E$ is

- injective if for any $x \neq y \in D$ $f(x) \neq f(y)$.
- surjective if for every $y \in E$ $\exists x \in D$ s.t. $f(x) = y$.
- bijective if it is both injective and surjective.

Examples . $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x$

f is injective: $x_1 \neq x_2$ then $f(x_1) = x_1 \neq x_2 = f(x_2)$

f is surjective: $\forall\, y \in \mathbb{R}$ $f(y) = y$.

f is bijective

• $f: \mathbb{R} \to [-1, 1]$ $f(x) = \sin(x)$

f is not injective: $\sin(0) = \sin(2\pi)$

but $0 \neq 2\pi$.

f is surjective $\forall\, y \in [-1, 1]$ $\exists\, x$ s.t $\sin(x) = y$.

- $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ $\quad f(x) = \sin x.$

- f is injective: If $x \neq y$ with

$-\frac{\pi}{2} \leq x, y \leq \frac{\pi}{2} \Rightarrow \sin(x) \neq \sin(y)$

- Not surjective since there is no x with $\sin(x) = 2$.

- $f : \mathbb{R} \to \mathbb{R}_{>0}$ $\quad f(x) = e^x$

f is injective since e^x is strictly monotone

$$e^x > e^y \text{ if } x > y$$

f is surjective: One way to guarantee this is by constructing $\log : \mathbb{R}_{>0} \to \mathbb{R}$ s.t. $\log(e^x) = x$.

Inverse functions

Let $f: D \to E$ be a function then we say that f is invertible if there exists $g: E \to D$ s.t.

$$g(f(x)) = x \quad \forall\, x \in D$$

$$f(g(y)) = y \quad \forall\, y \in E.$$

In this case we call g the inverse function to f and denote it by f^{-1}.

Theorem $f : E \to D$ is invertible if and only if f is bijective.

In this case the inverse f^{-1} is unique and defined by:

$$f^{-1}(y) := x \quad \text{s.t. } f(x) = y.$$

Such x exists by surjectivity and is unique by injectivity.

Example

- $f(x) = x^2 \quad f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$f^{-1}(x) = \sqrt{x}$ Check: 1) $f(f^{-1}(x)) = x$

2) $f^{-1}(f(x)) = x$

$\sqrt{x^2} = x$ if $x \geq 0$

$(\sqrt{x})^2 = x$ for $x \geq 0$

- $f(x) = 5x - 1$

$f^{-1}(y) = \frac{y+1}{5}$

Check: $f(f^{-1}(y)) =$

$f\left(\frac{y+1}{5}\right) = 5 \cdot \left(\frac{y+1}{5}\right) - 1$

$= y + 1 - 1 = y.$

Similarly $f^{-1}(f(x)) = x$

- Inverse trigonometric functions.

Here we restrict trigonometric functions on subsets of $\mathbb{R}$ in order to make them invertible.

Back to continuous functions on intervals.

Theorem 1.73 in Notes on Functions

Theorem Let $f \in C^0(I)$ then f is strictly monotone if and only if it is injective.

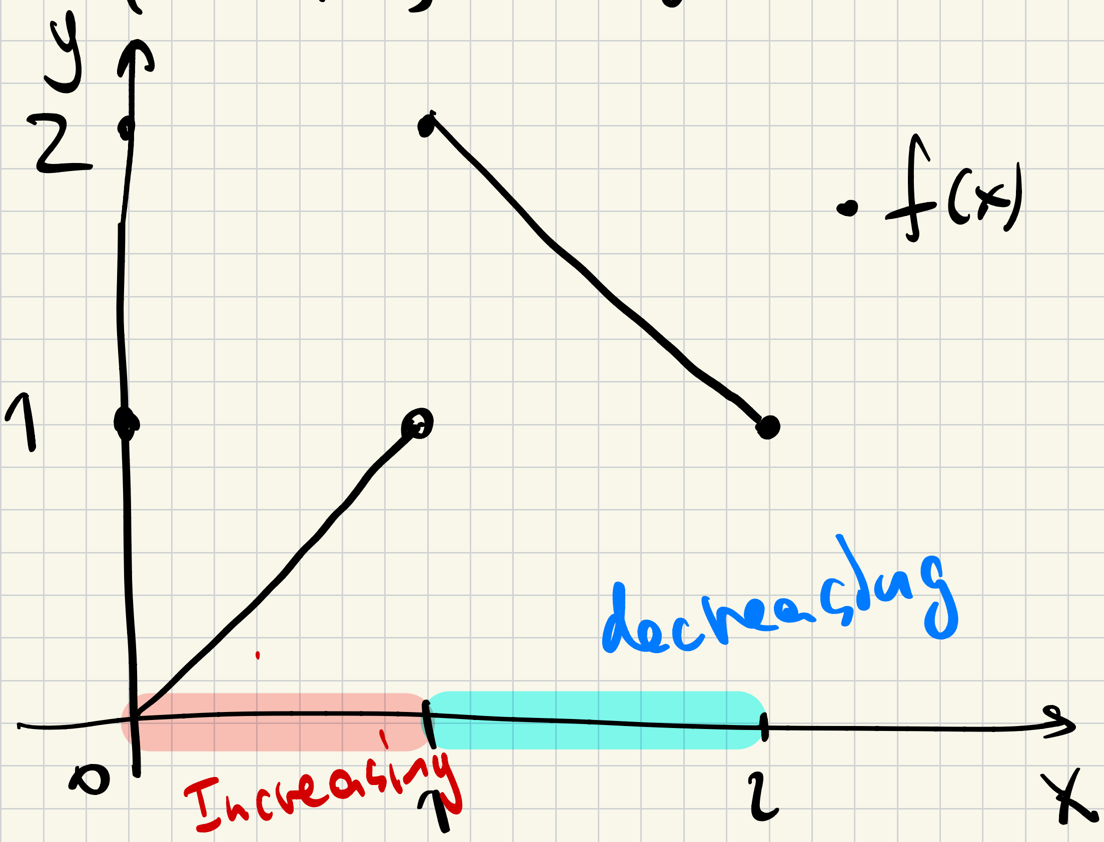
In general (for f not necessarily continuous)

we have strictly monotone $\Rightarrow$ injectivity

But $\not\Leftarrow$

Example let $f(x) = \begin{cases} x & 0 \le x < 1 \\ 3-x & 1 \le x \le 2 \end{cases}$

$f(x)$ is not strictly monotone



- $f(x)$ is injective:
  if $x \neq y$ then $f(x) \neq f(y)$.
- f is not continuous at $x = 1$.

Theorem 1.74

Theorem Let $f: I \to J$ be strictly monotone, continuous and surjective

then $f^{-1}: J \to I$ is also continuous.

Equivalent to injectivity since f is continuous.

Corollary All inverse trigonometric functions are continuous.

Differentiation.

(Chapter 2. in Notes on functions)

Definition 2.1 Let $f : E \to \mathbb{R}$ be a function and $x_0 \in E$.

1) We say f is differentiable at x_0 if the limit

$$\boxed{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}}$$

exists and finite.

We call the value of the limit the derivative of f at x_0 and denote it by $f'(x_0)$.

2) We say that f is differentiable if it is differentiable at every $x_0 \in E$

the derivatie of f denoted by f' is a function defined by

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Example $f(x) = x^n$ for $n \in \mathbb{N}_{\geq 0}$

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x^{n-1} + x^{n-2} x_0 + \dots + x_0^{n-1})}{(x - x_0)}$$

$$a^n - \cancel{a^{n-1} \cdot b} + \cancel{a^{n-1} \cdot b} - a^{n-2} \cdot b^2$$

we used identity: $(a-b)(a^{n-1} + a^{n-2} b + a^{n-3} b^2 + \dots + a b^{n-2} + b^{n-1})$

$$= a^n - b^n$$

$$f'(x) = \lim_{x \to x_0} \frac{(x - x_0)\left(x^{n-1} + x^{n-2} \cdot x_0 + \dots + x_0^{n-1}\right)}{(x - x_0)} =$$

$$= \lim_{x \to x_0} \underbrace{\left(x^{n-1} x_0^0 + x^{n-2} \cdot x_0 + \dots + x_0^{n-1}\right)}_{\substack{\text{Exactly } n \text{ summands} \\ \text{each evaluates to } x_0^{n-1} \\ \text{when } x = x_0}} = n \cdot x_0^{n-1}$$

$$\Rightarrow \boxed{(x^n)' = n \cdot x^{n-1}}$$

$\blacksquare$

Example

$f(x) = \sin(x)$

trig. identity

$$f'(x) = \lim_{x \to x_0} \frac{\sin(x) - \sin(x_0)}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{2 \sin\left(\frac{x - x_0}{2}\right) \cdot \cos\left(\frac{x + x_0}{2}\right)}{x - x_0} =$$

$$= \lim_{x \to x_0} \left(\frac{\sin\left(\frac{x - x_0}{2}\right)}{\frac{x - x_0}{2}} \cdot \cos\left(\frac{x + x_0}{2}\right) \right)$$

$$= \lim_{x \to x_0} \left(\underbrace{\frac{\sin\left(\frac{x - x_0}{2}\right)}{\frac{x - x_0}{2}}}_{\text{converges to } 1} \cdot \underbrace{\cos\left(\frac{x + x_0}{2}\right)}_{\to \cos\left(\frac{x_0 + x_0}{2}\right) = \cos(x_0)} \right)$$

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

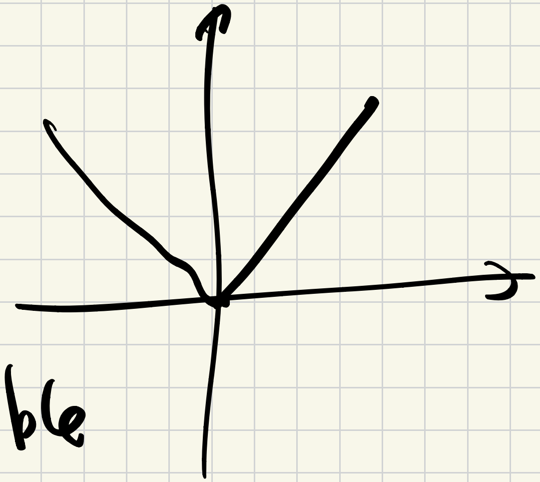
since cos is cont.

so we can plug in x_0 into it.

$$\Rightarrow \boxed{(\sin x)' = \cos(x)}$$

∎

Example $f(x) = |x| \quad x_0 = 0$



Claim f is not differentible at $x_0 = 0$.

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{|x|-|0|}{x-0} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

$$\Rightarrow \lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = 1 \quad \text{but} \quad \lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = -1$$

$\Rightarrow \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exists $\Rightarrow f$ is not diff. at 0.

Proposition 2.8 If $f: E \to \mathbb{R}$ is differentiable at x_0 then it is continuous at x_0.

When checking differentiability first check continuity.

Computing derivatives

Propositions 2.11, 2.13, 2.15, 2.18

Let $f, g : E \to \mathbb{R}$ which are differentiable at x_0 then.

1) $f+g$ is differentiable and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

2) $f \cdot g$ is differentiable at x_0

and $$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$$

Called Leibniz rule for derivation.

3) if $g(x_0) \neq 0$ then f/g is differentiable and

$$\left(f/g\right)'(x_0) = \frac{f' \cdot g(x_0) - f \cdot g'(x_0)}{g^2(x_0)}$$

in particular, $\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$